MA4271 Differential Geometry of Curves and Surfaces

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Reference books:

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Contents

Chapter 1
Curves

1.1
Parametrized Curves

Let $\mathbb{R}^3$ denote the set of triples (x,y,z) of real numbers. Our goal is to characterise certain subsets of $\mathbb{R}^3$ (to be called curves) that are, in a certain sense, one-dimensional and to which the methods of Differential Calculus can be applied. A natural way of defining such subsets is through differentiable functions. Recall that a real function of a real variable is differentiable, or smooth, if it has at all points derivatives of all orders.

Definition 1.1 (parametrised differentiable curve)**.** A parametrised differentiable curve is a differentiable map

$$\alpha : I \to \mathbb{R}^3 \text{ of an open interval } I = (a,b) \quad \text{of the real line } \mathbb{R} \quad \text{into } \mathbb{R}^3.$$

The word differentiable in Definition 1.1 means that α is a correspondence which maps each $t \in I$ into a point $\alpha(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$ in such a way that the functions x, y, z are differentiable. The variable t is called the parameter of the curve, and the word interval is taken in a generalised sense so that we exclude the cases $\pm\infty$.

Given a parametrised differentiable curve $\alpha(t) = (x(t), y(t), z(t))$, the derivative $\alpha'(t) = (x'(t), y'(t), z'(t))$ is the tangent vector at t. The image set $\alpha(I) \subset \mathbb{R}^3$ is called the trace of α.

Example 1.1. The curve $\alpha(t) = (a\cos t, a\sin t, bt)$ traces a helix on the cylinder $x^2 + y^2 = a^2$.

Example 1.2. The curve $\alpha(t) = (t^3, t^2)$ has zero velocity at $t = 0$.

Example 1.3. The curve $\alpha(t) = (t^3 - 4t, t^2 - 4)$ is not injective as $\alpha(2) = \alpha(-2) = (0,0)$.

Example 1.4. The map $\alpha(t) = (t, |t|)$ is not differentiable at $t = 0$, hence not a differentiable curve.

Example 1.5. $\alpha(t) = (\cos t, \sin t)$ and $\beta(t) = (\cos 2t, \sin 2t)$ both trace the circle $x^2 + y^2 = 1$, but with different velocities.

We now recall the inner product. For vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, the norm is $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$, and the inner product is defined by $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$. It satisfies the usual properties for which we will not discuss. Using standard basis vectors, the dot product becomes $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$. Also, if $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are differentiable curves, then by the product rule, we have

$$\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t).$$

1.2
Arc Length

Let $\alpha : I \to \mathbb{R}^3$ be a parametrised differentiable curve. For each $t \in I$ where $\alpha'(t) \neq 0$, there is a well-defined straight line, which contains the point $\alpha(t)$ and the vector $\alpha'(t)$. This line is called the tangent line to α at t. For the study of the Differential Geometry of a curve, it is essential that there exists such a tangent line at every point. Therefore, we call any point t where $\alpha'(t) = 0$ a singular point of α and restrict our attention to curves without singular points.

Definition 1.2 (regular curve)**.** A parametrized differentiable curve $\alpha : I \to \mathbb{R}^3$ is said to be regular if $\alpha'(t) \neq 0$ for all $t \in I$.

From now on we shall consider only regular parametrized differentiable curves (and, for convenience, shall usually omit the word differentiable). Recall from MA2002 that given $t_0 \in I$, the arc length of a regular parametrized curve $\alpha : I \to \mathbb{R}^3$, from the point t_0, is

$$s(t) = \int_{t_0}^{t} |\alpha'(t)| dt \quad \text{where} \quad |\alpha'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

is the length of the vector $\alpha'(t)$. Since $\alpha'(t) \neq 0$, the arc length s is a differentiable function of t and $\frac{ds}{dt} = |\alpha'(t)|$.

It can happen that the parameter t is already the arc length measured from some point. In this case, $\frac{ds}{dt} = 1 = |\alpha'(t)|$; that is, the velocity vector has constant length equal to 1. Conversely, if $|\alpha'(t)| = 1$, then

$$s = \int_{t_0}^{t} dt = t - t_0,$$

i.e. t is the arc length of α measured from some point. To simplify our exposition, we shall restrict ourselves to curves parametrised by arc length; we will later see that this restriction is not essential. In general, it is not necessary to mention the origin of the arc length s, since most concepts are defined only in terms of the derivatives of $\alpha(s)$. It is convenient to set still another convention. Given the curve α parametrised by arc length $s \in (a,b)$, we may consider the curve β defined in $(-b,-a)$ by $\beta(-s) = \alpha(s)$, which has the same trace as the first one but is described in the opposite direction. We say, then, that these two curves differ by a change of orientation.

Now, let $\alpha : I = (a,b) \to \mathbb{R}^3$ be a curve parametrised by arc length s. Since the tangent vector $\alpha'(s)$ has unit length, the norm $|\alpha''(s)|$ of the second derivative measures the rate of change of the angle which neighbouring tangents make with the tangent at s. As such, $|\alpha''(s)|$ gives a measure of how rapidly the curve pulls away from the tangent line at s in a neighbourhood of s. Hence, we have the following definition on curvature (Definition 1.3):

Definition 1.3 (curvature)**.** Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrised by arc length $s \in I$. The number

$$|\alpha''(s)| = k(s) \quad \text{is} \quad \text{called the curvature of } \alpha \text{ at } s.$$

Note that if α is a straight line, then $\alpha(s) = us + v$, where u and v are constant vectors, so $k = 0$. Conversely, if $k = |\alpha''(s)| = 0$, then by integration, we have $\alpha(s) = us + v$, so the curve is a straight line.

Note that by a change of orientation, the tangent vector changes its direction, i.e. if $\beta(-s) = \alpha(s)$, then

$$\frac{d\beta}{d(-s)}(-s) = -\frac{d\alpha}{ds}(s).$$

So, $\alpha''(s)$ and the curvature remain invariant under a change of orientation.

At points where $k(s) \neq 0$, a unit vector $n(s)$ in the direction $\alpha''(s)$ is well defined by the equation

$$\alpha''(s) = k(s)n(s).$$

Moreover, $\alpha''(s)$ is normal to $\alpha'(s)$, because by differentiating $\alpha'(s) \cdot \alpha'(s) = 1$ we obtain $\alpha''(s) \cdot \alpha'(s) = 0$. Thus, $n(s)$ is normal to $\alpha'(s)$ and is called the normal vector at s. The plane determined by the unit tangent and normal vectors, $\alpha'(s)$ and $n(s)$, is called the osculating plane at s.

At points where $k(s) = 0$, the normal vector (and therefore the osculating plane) is not defined. To proceed with the local analysis of curves, we need, in an essential way, the osculating plane. It is therefore convenient to say that $s \in I$ is a singular point of order 1 if $\alpha''(s) = 0$. In this context, the points where $\alpha'(s) = 0$ are called singular points of order 0.

In what follows, we shall restrict ourselves to curves parametrised by arc length without singular points of order 1. We shall denote by $t(s) = \alpha'(s)$ the unit tangent vector of α at s. Thus,

$$t'(s) = k(s)n(s).$$

The unit vector $b(s) = t(s) \times n(s)$ is normal to the osculating plane and will be called the binormal vector at s. Since $b(s)$ is a unit vector, the length $|b'(s)|$ measures the rate of change of the neighbouring osculating planes with the osculating plane at s; that is, $|b'(s)|$ measures how rapidly the curve pulls away from the osculating plane at s, in a neighbourhood of s.

To compute $b'(s)$, we observe that, on the one hand, $b'(s)$ is normal to $b(s)$ and that, on the other hand,

$$b'(s) = t'(s) \times n(s) + t(s) \times n'(s) = t(s) \times n'(s),$$

that is, $b'(s)$ is normal to $t(s)$. It follows that $b'(s)$ is parallel to $n(s)$, and we may write

$$b'(s) = \tau(s)n(s)$$

for some function $\tau(s)$, where τ denotes torsion (Definition 1.4).

Definition 1.4 (torsion). Let $\alpha\colon I \to \mathbb{R}^3$ be a curve parametrised by arc length s such that $\alpha''(s) \neq 0$, $s \in I$. The number $\tau(s)$ defined by $b'(s) = \tau(s)n(s)$ is called the torsion of α at s.

If α is a plane curve (that is, $\alpha(I)$ is contained in a plane), then the plane of the curve agrees with the osculating plane; hence, $\tau \equiv 0$. Conversely, if $\tau \equiv 0$ (and $k \neq 0$), we have that $b(s)$ being some constant, and therefore

$$(\alpha(s) \cdot b_0)' = \alpha'(s) \cdot b_0 = 0.$$

It follows that $\alpha(s) \cdot b_0$ is a constant. Hence, $\alpha(s)$ is contained in a plane normal to b_0. The condition that $k \neq 0$ everywhere is essential here. Notice that by changing orientation, the binormal vector changes sign, since $b = t \times n$. It follows that $b'(s)$, and, therefore, the torsion, remain invariant under a change of orientation.

We now summarise our position. To each value of the parameter s, we have associated three orthogonal unit vectors $t(s)$, $n(s)$, $b(s)$. The trihedron thus formed is referred to as the Frenet trihedron at s. The derivatives

$$t'(s) = kn \quad \text{and} \quad b'(s) = \tau n$$

of the vectors $t(s)$ and $b(s)$, when expressed in terms of the basis $\{t, n, b\}$, yield geometrical entities (curvature k and torsion τ) which give us information about the behaviour of α in a neighbourhood of s.

The search for other local geometrical entities would lead us to compute $n'(s)$. However, since $n = b \times t$, we have

$$n'(s) = b'(s) \times t(s) + b(s) \times t'(s) = -\tau b - kt,$$

and we obtain again the curvature and the torsion.

Proposition 1.1 (Frenet formulae). We have the following:

$$t' = kn \quad n' = -kt - \tau b \quad b' = \tau n$$

The tb plane is normal plane. The lines which contain $n(s)$ and $b(s)$ and pass through $\alpha(s)$ are called the principal normal and the binormal respectively. The inverse $R = 1/k$ of the curvature is called the radius of curvature at s. Physically, we can think of a curve in $\mathbb{R}^3$ as being obtained from a straight line by bending (curvature) and twisting (torsion). After reflecting on this construction, we are led to the following fundamental statement (Theorem 1.1).

Theorem 1.1 (fundamental theorem of the local theory of curves). Given differentiable functions $k(s) > 0$ and $\tau(s)$, $s \in I$, there exists a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ such that s is the arc length, $k(s)$ is the curvature, and $\tau(s)$ is the torsion of α. Moreover, any other curve $\bar{\alpha}$, satisfying the same conditions, differs from α by a rigid motion; that is, there exists an orthogonal linear map ρ of $\mathbb{R}^3$, with positive determinant, and a vector c such that

$$\bar{\alpha} = \rho \circ \alpha + c.$$

Theorem 1.1 tells us that a space curve is uniquely determined (up to rigid motion) by its curvature $k(s)$ and torsion $\tau(s)$, assuming s is arc length. We shall go through a few classical examples that illustrate this theorem.

Example 1.6 (straight line). Let $k(s) = 0$, which automatically implies $\tau(s) = 0$ (since torsion is only defined when $k(s) > 0$). The unique curve with zero curvature is a straight line. For instance, $\alpha(s) = (s, 0, 0)$. Any other line with constant speed (unit-speed in arc length) is just a rotated and translated version of this one.

Example 1.7 (circle in a plane). Let $k(s) = \kappa > 0$ (constant), and $\tau(s) = 0$. This describes a circle in a plane. A typical parametrisation with arc length s is

$$\alpha(s) = \left(\frac{1}{\kappa} \cos(\kappa s),\ \frac{1}{\kappa} \sin(\kappa s),\ 0 \right).$$

The curvature is constant and the torsion is zero, so the curve lies entirely in a plane.

Chapter 2
Regular Surfaces

2.1 Regular Surfaces

We shall introduce the notion of a regular surface in $\mathbb{R}^3$. A regular surface in $\mathbb{R}^3$ is obtained by taking pieces of a plane, deforming them, and arranging them in such a way that the resulting figure has no sharp points, edges, or self-intersections and so that it makes sense to speak of a tangent plane at points of the figure. The idea is to define a set that is in a certain sense two-dimensional and that is smooth enough so that the usual notions of Calculus can be extended to it.

Definition 2.1 (regular surface). A subset $S \subseteq \mathbb{R}^3$ is a regular surface if for each $\mathbf{p} \in S$, there exists a neighbourhood V in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subseteq \mathbb{R}^2$ onto $V \cap S \subseteq \mathbb{R}^3$ such that the following hold:

- **(i)** $\mathbf{x}$ is differentiable, i.e. its component functions have continuous partial derivatives of all orders on U
- **(ii)** $\mathbf{x}$ is a homeomorphism
- **(iii)** **Regularity condition:** for each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective

In Definition 2.1, the map $\mathbf{x}$ is called a parametrisation or a system of local coordinates in a neighbourhood of p. The neighbourhood $V \cap S$ of $\mathbf{p}$ in S is called a coordinate neighbourhood. As for condition **(iii)**, we give some explanation. We compue the matrix of the linear map $d\mathbf{x}_q$ in the canonical bases $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ of $\mathbb{R}^2$ with the coordinates (u,v) and $\mathbf{f}_1 = (1,0,0)$, $\mathbf{f}_2 = (0,1,0)$ and $\mathbf{f}_3 = (0,0,1)$ of $\mathbb{R}^3$ with coordinates (x,y,z).

Let $\mathbf{q} = (u_0, v_0)$. Then, the vector $\mathbf{e}_1$ is tagnent to the curve $u \to (u, v_0)$ whose image under $\mathbf{x}$ is the curve

$$u \to (x(u,v_0), y(u,v_0), z(u,v_0)).$$

This image curve lies on S and has at $\mathbf{x}(q)$ the tangent vector

$$\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \frac{\partial \mathbf{x}}{\partial u}.$$

By the definition of the differential, we have

$$d\mathbf{x}_q(\mathbf{e}_1)\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \frac{\partial \mathbf{x}}{\partial u}.$$

Similarly, using the coordinate curve $u = u_0$, we obtain

$$d\mathbf{x}_q(e_2) = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = \frac{\partial \mathbf{x}}{\partial v}.$$

Thus, the matrix of the linear map $d\mathbf{x}_q$ in the referred basis is

$$d\mathbf{x}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

As such, condition **(iii)** of Definition 2.1 can be expressed by requiring the two column vectors of the matrix $d\mathbf{x}_q$ to be linearly independent.

Example 2.1. The unit sphere

$$S^2 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\right\} \quad \text{is a regular surface.}$$

To see why, we first verify that the map $\mathbf{x}_1 : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathbf{x}_1(x,y) = (x,y,\sqrt{1-(x^2+y^2)}) \quad \text{where } (x,y) \in U,$$

is a parametrisation of S^2. Here, $\mathbb{R}^2$ denotes the 2-dimensional Euclidean space, i.e. we can set $z=0$, and U denotes the set of $(x,y) \in \mathbb{R}^2$ such that $x^2+y^2<1$. One observes that $\mathbf{x}_1(U)$ is the open part of S^2 above the xy-plane. Since $x^2+y^2<1$, the function $\sqrt{1-(x^2+y^2)}$ has continuous partial derivatives of all orders. So, $\mathbf{x}_1$ is differentiable and condition **(i)** of Definition 2.1 holds.

Next, **(iii)** is easily verified since

$$\frac{\partial(x,y)}{\partial(x,y)} = 1.$$

Lastly, we verify **(ii)**, i.e. $\mathbf{x}_1$ is a homeomorphism. Note that $\mathbf{x}_1$ is injective and $\mathbf{x}_1^{-1}$ is the restriction of the continuous projection $\pi(x,y,z) = (x,y)$ to the set $\mathbf{x}_1(U)$. Thus, $\mathbf{x}_1^{-1}$ is continuous on $\mathbf{x}_1(U)$. Hence, $\mathbf{x}_1$ is a homeomorphism.

Note that the whole sphere can be completely covered by six maps. First, define

$$\mathbf{x}_2 : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3 \quad \text{where} \quad \mathbf{x}_2(x,y) = (x,y,-\sqrt{1-(x^2+y^2)}).$$

One checks that $\mathbf{x}_2$ is a parametrisation and observe that $\mathbf{x}_1(U) \cup \mathbf{x}_2(U)$ covers S^2 minus the equator. Then, using the xz- and yz-planes, one can define similar parametrisations $\mathbf{x}_3,\ldots,\mathbf{x}_6$. So, together with $\mathbf{x}_1$ and $\mathbf{x}_2$, these maps cover S^2 completely and this shows that S^2 is a regular surface.

For most applications, it is convenient to relate parametrisations to the geographical coordinates on S^2. Recall the notion of spherical coordinates. Let

$$V = \{(\theta,\phi) : 0<\theta<2\pi \text{ and } 0<\phi<\pi\}$$

and let

$$\mathbf{x} : V \to \mathbb{R}^3 \quad \text{where} \quad \mathbf{x}(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi).$$

Then, $\mathbf{x}(V) \subseteq S^2$. We justify that $\mathbf{x}$ is a parametrisation of S^2. Note that ϕ is usually called the coaltitude and θ the longitude. It is clear that the functions $\cos\theta\sin\phi$, $\sin\theta\sin\phi$, and $\cos\phi$ have continuous partial derivatives on all orders so $\mathbf{x}$ is differentiable. Moreover, in order for the Jacobian determinants to vanish simultaenously, we must have $\sin\phi = 0$[†] This does not occur in V so **(i)** and **(iii)** of Definition 2.1 are satisfied.

We then observe that given $(x,y,z) \in S^2 \setminus C$, where C denotes the semicircle

$$C = \left\{(x,y,z) \in S^2 : y=0, x \geq 0\right\},$$

ϕ is uniquely determined by $\phi = \cos^{-1} z$ since $0<\phi<\pi$. By knowing ϕ, we find $\sin\theta$ and $\cos\theta$ using $x = \cos\theta\sin\phi$ and $y = \sin\theta\sin\phi$, and determines θ uniquely, where $0<\theta<2\pi$. It follows that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1}$. To complete the verification of **(ii)**, we should prove that $\mathbf{x}^{-1}$ is continuous. We omit the details.

Based on Example 2.1, we see that deciding whether a given subset of $\mathbb{R}^3$ is a regular surface directly from the definition may be quite tiresome. We now present two propositions which will simplify the task.

[†] Work out the Jacobian determinants and you would see that $\sin\phi$ is a common term.

Proposition 2.1. If $f : U \to \mathbb{R}$ is a differentiable function on an open set $U \subseteq \mathbb{R}^2$, then the graph of f is a regular surface.

Definition 2.2. Given a differentiable map $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ defined on an open set U of $\mathbb{R}^n$, we say that $p \in U$ is a critical point of F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not a surjective map. The image $F(p) \subseteq \mathbb{R}^m$ of a critical point is a critical value of F; a point of $\mathbb{R}^m$ which is not a critical value is a regular value of F.

Proposition 2.2. If $f : U \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.

Example 2.2. We now relook at a slightly modified version of Example 2.1. Let

$$f : \mathbb{R}^3 \to \mathbb{R} \quad \text{where} \quad f(x,y,z) = x^2 + y^2 + z^2.$$

We want to determine for which values $a \in \mathbb{R}$ the level set $f^{-1}(a)$ is a regular surface. First, compute the gradient (which is the differential), i.e. $\nabla f(x,y,z) = (2x, 2y, 2z)$. This gradient is zero only at the point $(0,0,0)$, so the only critical point is at the origin. Thus, the only critical value is $f(0,0,0) = 0$. Hence, any $a > 0$ is a regular value of f. By the proposition, for each regular value $a > 0$, the level set

$$f^{-1}(a) = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = a\}$$

is a regular surface in $\mathbb{R}^3$ — in fact, it is the sphere of radius $\sqrt{a}$ centred at the origin. We can generalise this example of an ellipsoid.

The examples of regular surfaces presented thus far have been connected subsets of $\mathbb{R}^3$. Intuitively, a surface $S \subseteq \mathbb{R}^3$ is said to be connected if any two of its points can be joined by a continuous curve in S. In Definition 2.1, we made no restrictions on the connectedness of the surfaces, and Example 2.3 shows that the regular surfaces given by Proposition 2.2 may not be connected.

Example 2.3. The hyperboloid of two sheets $x^2 + y^2 - z^2 = -1$ is a regular surface since it is given by $S = f^{-1}(0)$, where 0 is a regular value of $f(x,y,z) = x^2 + y^2 - z^2 + 1$. Note that the surface S is not connected.

Example 2.4. The torus $\mathbb{T}$ is a surface generated by rotating a circle S^1 of radius r about a straight line belonging to the plane of the circle and at a distance $a > r$ away from the centre of the circle. Let S^1 be the circle in the yz-plane with its centre at $(0,a,0)$. Then, S^1 is given by the equation $(y-a)^2 + z^2 = r^2$, and the points of the figure $\mathbb{T}$ obtained by rotating this circle about the z-axis satisfy the equation

$$z^2 = r^2 - \left(\sqrt{x^2+y^2} - a\right)^2.$$

Hence, $\mathbb{T}$ is the inverse image of r^2 by

$$f(x,y,z) = z^2 + \left(\sqrt{x^2+y^2} - a\right)^2.$$

For $(x,y) \neq (0,0)$, f is differentiable. Also, the partial derivatives are continuous so r^2 is a regular value of f. It follows that the torus $\mathbb{T}$ is a regular surface.

Proposition 2.3. Let $S \subseteq \mathbb{R}^3$ be a regular surface and $p \in S$. Then, there exists a neighbourhood V of p in S such that V is the graph of a differentiable function which has one of the following three forms:

$$z = f(x,y) \quad y = g(x,z) \quad x = h(y,z)$$

Proposition 2.4. Let $p \in S$ be a point of a regular surface S and let $\mathbf{x} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a map with $p \in \mathbf{x}(U) \subseteq S$ such that **(i)** and **(iii)** of Definition 2.1 hold. Assume that $\mathbf{x}$ is injective. Then, $\mathbf{x}^{-1}$ is continuous.

Example 2.5. The one-sheeted cone C given by $z = \sqrt{x^2+y^2}$ where $(x,y) \in \mathbb{R}^2$ is not a regular surface. Observe that we cannot conclude this from the fact alone that the natural parametrisation $(x,y) \mapsto (x,y,\sqrt{x^2+y^2})$ is not differentiable because there could be other parametrisations satisfying Definition 2.1.

To show that the one-sheeted cone is not a regular surface, it would be, in a neighbourhood of $(0,0,0) \in C$, the graph of a differentiable function having one of the following three forms by Proposition 2.3:

$$y = h(x,z) \quad x = g(y,z) \quad z = f(x,y)$$

The first two forms can be discarded since the projections of C over the xz- and yz-planes are not injective. The last form would have to agree in a neighbourhood of the origin, with $z = \sqrt{x^2+y^2}$. Since $z = \sqrt{x^2+y^2}$ is not differentiable at the origin, we conclude that this is impossible.

2.2 Change of Parameters

Differential Geometry is concerned with the local properties of surfaces which depend on their behaviour in a neighbourhood of a point. According to the definition of a regular surface (Definition 2.1), each point p lies in a coordinate neighbourhood, enabling the expression of functions locally in terms of coordinates.

To define what it means for a function $f : S \to \mathbb{R}$ to be differentiable at a point $p \in S$, one chooses a coordinate chart (u,v) around p, and requires that the function f expressed in these coordinates has continuous partial derivatives of all orders. However, a point may lie in multiple overlapping coordinate neighbourhoods. For the notion of differentiability to be well-defined, it must be independent of the chosen parametrisation. This is guaranteed if the change of coordinates between two parametrisations is a diffeomorphism.

Proposition 2.5 (change of parameters)**.** Let p be a point on a regular surface S, and let

$$\mathbf{x} : U \subseteq \mathbb{R}^2 \to S \quad \text{and} \quad \mathbf{y} : V \subseteq \mathbb{R}^2 \to S$$

be two parametrisations such that $p \in \mathbf{x}(U) \cap \mathbf{y}(V) = W$. Define the change of coordinates map

$$h = \mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W).$$

Then h is a diffeomorphism, i.e. h is differentiable and has a differentiable inverse h^{-1}.

In other words, let the parametrisations be given by

$$\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)) \quad (u,v) \in U,$$
$$\mathbf{y}(\xi,\eta) = (x(\xi,\eta), y(\xi,\eta), z(\xi,\eta)) \quad (\xi,\eta) \in V.$$

Then, the change of variables h takes the form

$$u = u(\xi,\eta) \quad v = v(\xi,\eta) \quad (\xi,\eta) \in \mathbf{y}^{-1}(W),$$

with the inverse given by

$$\xi = \xi(u,v) \quad \eta = \eta(u,v) \quad (u,v) \in \mathbf{x}^{-1}(W).$$

Both transformations have partial derivatives of all orders, and their Jacobians are non-zero everywhere. Specifically,

$$\frac{\partial(u,v)}{\partial(\xi,\eta)} \cdot \frac{\partial(\xi,\eta)}{\partial(u,v)} = 1,$$

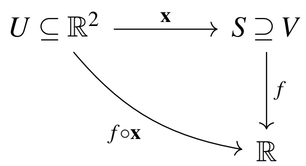
so each Jacobian determinant is non-zero.

Definition 2.3 (differentiability). Let $f : V \subseteq S \to \mathbb{R}$ be a function defined on an open subset V of a regular surface S. Then f is said to be differentiable at $p \in V$ if, for some parametrization $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ with $p \in \mathbf{x}(U) \subseteq V$, the composition

$$f \circ \mathbf{x} : U \to \mathbb{R}$$

is differentiable at $\mathbf{x}^{-1}(p)$. The function f is differentiable on V if it is differentiable at all $p \in V$.

We illustrate the composition in Definition 2.3 as follows:



Definition 2.3 does not depend on the choice of parametrisation. If $\mathbf{y} : V \to S$ is another parametrisation with $p \in \mathbf{y}(V)$, and if $h = \mathbf{x}^{-1} \circ \mathbf{y}$, then

$$f \circ \mathbf{y} = f \circ \mathbf{x} \circ h,$$

which is differentiable since both $f \circ \mathbf{x}$ and h are differentiable. Hence, the definition is coordinate-invariant.

Example 2.6. Let S be a regular surface and $V \subseteq \mathbb{R}^3$ be an open set such that $S \subseteq V$. Let $f : V \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. Then, the restriction of f to S is a differentiable function on S. In fact, for any $p \in S$ and any parametrisation $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ in p, the function $f \circ \mathbf{x} : U \to \mathbb{R}$ is differentiable. In particular, the following functions are differentiable:

(i) The height function relative to a unit vector $\mathbf{v} \in \mathbb{R}^3$, where $h : S \to \mathbb{R}$, given by $h(\mathbf{p}) = \mathbf{p} \cdot \mathbf{v}$, $\mathbf{p} \in S$, where the dot product denotes the usual inner product in $\mathbb{R}^3$. Geometrically, $h(\mathbf{p})$ is the height of $\mathbf{p} \in S$ relative to a plane normal to $\mathbf{v}$ and passing through the origin of $\mathbb{R}^3$.

(ii) The square of the distance from a fixed point $\mathbf{p}_0 \in \mathbb{R}^3$, denoted by $f(\mathbf{p}) = |\mathbf{p} - \mathbf{p}_0|^2$, $\mathbf{p} \in S$, is differentiable. The need for taking the square comes from the fact that the distance $|\mathbf{p} - \mathbf{p}_0|$ is not differentiable at $\mathbf{p} = \mathbf{p}_0$.

Definition 2.4 (differentiability). A continuous map $\varphi : V_1 \subseteq S_1 \to S_2$ from an open set of a regular surface S_1 to another regular surface S_2 is said to be differentiable at $p \in V_1$ if, for parametrisations

$$\mathbf{x}_1 : U_1 \subseteq \mathbb{R}^2 \to S_1 \quad \text{and} \quad \mathbf{x}_2 : U_2 \subseteq \mathbb{R}^2 \to S_2$$

with $p \in \mathbf{x}_1(U_1)$ and $\varphi(\mathbf{x}_1(U_1)) \subseteq \mathbf{x}_2(U_2)$, the map

$$\mathbf{x}_2^{-1} \circ \varphi \circ \mathbf{x}_1 : U_1 \to U_2 \quad \text{is} \quad \text{differentiable at } q = \mathbf{x}_1^{-1}(p).$$

Definition 2.4 can be interpreted as follows too. If $\varphi(u_1,v_1) = (\varphi_1(u_1,v_1), \varphi_2(u_1,v_1))$, then φ is differentiable if all partial derivatives of φ_1 and φ_2 exist and are continuous.

Definition 2.5 (diffeomorphism). Two regular surfaces S_1 and S_2 are diffeomorphic if there exists

$$\text{a differentiable bijection } \varphi : S_1 \to S_2 \quad \text{with} \quad \text{a differentiable inverse } \varphi^{-1} : S_2 \to S_1.$$

This relation plays the role of isomorphism in Differentiable Geometry.

Example 2.7. Let $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ be a parametrisation. Then $\mathbf{x}^{-1} : \mathbf{x}(U) \to \mathbb{R}^2$ is differentiable, and for any other parametrisation $\mathbf{y} : V \subseteq \mathbb{R}^2 \to S$, the map

$$\mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W) \quad \text{where} \quad W = \mathbf{x}(U) \cap \mathbf{y}(V)$$

is a diffeomorphism.

Example 2.8. Let S_1, S_2 be regular surfaces in an open set $V \subseteq \mathbb{R}^3$, and let $\varphi : V \to \mathbb{R}^3$ be a differentiable map such that $\varphi(S_1) \subseteq S_2$. Then the restriction $\varphi\,|_{S_1} : S_1 \to S_2$ is differentiable. For example, if S is symmetric about a plane, then $\sigma(x,y,z) = (x,y,-z)$ restricts to a differentiable map on S; If $R_{z,\theta}$ is rotation about the z-axis, then $R_{z,\theta}\,|_S$ is differentiable; if $\varphi(x,y,z) = (ax,by,cz)$, then $\varphi\,|_{S^2}$ maps the unit sphere to an ellipsoid.

Definition 2.6 (regular curve)**.** A regular curve $C \subseteq \mathbb{R}^3$ is one where for every $p \in C$, there exists a neighbourhood $V \subseteq \mathbb{R}^3$ and a differentiable homeomorphism $\alpha : I \subseteq \mathbb{R} \to V \cap C$ with $d\alpha$ one-to-one.

Properties defined via parametrisations, like arc length, curvature, and torsion, are thus local properties.

Example 2.9 (surfaces of revolution)**.** Let a regular plane curve C be parametrized as $x = f(v)$ and $z = g(v)$, with $f(v) > 0$, and define a surface by rotating C about the z-axis. Define

$$\mathbf{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v)).$$

Then, $\mathbf{x}$ is a regular parametrisation of the surface of revolution. To show that $\mathbf{x}$ is a homeomorphism, note that $(f(v))^2 = x^2 + y^2$, so v is uniquely determined. Also,

$$u = 2\tan^{-1}\left(\frac{y}{x + \sqrt{x^2+y^2}}\right) \quad \text{or} \quad u = 2\cot^{-1}\left(\frac{y}{-x + \sqrt{x^2+y^2}}\right) \text{ if } u \approx \pi$$

This confirms $\mathbf{x}^{-1}$ is continuous and hence $\mathbf{x}$ is a diffeomorphism onto its image.

2.3 The Tangent Plane and the Differential of a Map

Here, we show that condition **(iii)** in the definition of a regular surface S (Definition 2.1) guarantees that for every point $p \in S$, the set of tangent vectors to the parametrised curves in S passing through p constitutes a plane. By a tangent vector to S at a point $p \in S$, we mean the tangent vector $\alpha'(0)$ of a differentiable parametrised curve

$$\alpha : (-\varepsilon, \varepsilon) \to S \quad \text{with} \quad \alpha(0) = p.$$

Proposition 2.6. Let $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ be a parametrisation of a regular surface S, and let $q \in U$. Then the image of the differential

$$d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$$

is a 2-dimensional subspace of $\mathbb{R}^3$ that coincides with the set of all tangent vectors to S at the point $\mathbf{x}(q)$.

By Proposition 2.6, the plane $d\mathbf{x}_q(\mathbb{R}^2)$, which passes through $\mathbf{x}(q) = p$, does not depend on the particular choice of parametrisation $\mathbf{x}$. This plane will be called the tangent plane to S at p and is denoted by $T_p(S)$. The choice of parametrisation $\mathbf{x}$ determines a basis $\{\mathbf{x}_u(q), \mathbf{x}_v(q)\}$ of $T_p(S)$, called the basis associated to $\mathbf{x}$. Sometimes, it is convenient to write

$$\frac{\partial \mathbf{x}}{\partial u} = \mathbf{x}_u \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial v} = \mathbf{x}_v.$$

Example 2.10. We take the standard (elliptic) paraboloid

$$S = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2\}$$

and parametrise it by $\mathbf{x}(u,v) = \big(u, v, u^2+v^2\big)$, where $(u,v) \in U = \mathbb{R}^2$. Fix a point $q = (u_0, v_0) \in U$. Then, $\mathbf{x}(u_0,v_0) = \big(u_0, v_0, u_0^2+v_0^2\big) \in S$. We compute the partial derivatives of $\mathbf{x}$ at (u_0,v_0), so

$$\mathbf{x}_u(u,v) = \Big(1, 0, 2u\Big) \quad \text{and} \quad \mathbf{x}_v(u,v) = \Big(0, 1, 2v\Big).$$

In particular, at $q = (u_0,v_0)$ these become

$$\mathbf{x}_u(u_0,v_0) = \big(1, 0, 2u_0\big) \quad \text{and} \quad \mathbf{x}_v(u_0,v_0) = \big(0, 1, 2v_0\big).$$

By definition, the differential

$$d\mathbf{x}_{(u_0,v_0)} : \mathbb{R}^2 \to \mathbb{R}^3$$

sends a tangent-vector $(\delta u, \delta v) \in T_{(u_0,v_0)}\mathbb{R}^2 \simeq \mathbb{R}^2$ to

$$d\mathbf{x}_{(u_0,v_0)}\big(\delta u, \delta v\big) = \delta u \mathbf{x}_u(u_0,v_0) + \delta v \mathbf{x}_v(u_0,v_0) = \big(\delta u, \delta v, 2u_0\delta u + 2v_0 \delta v\big).$$

Thus,

$$\operatorname{Im}\big(d\mathbf{x}_{(u_0,v_0)}\big) = \operatorname{span}\Big\{\mathbf{x}_u(u_0,v_0), \mathbf{x}_v(u_0,v_0)\Big\} = \operatorname{span}\big\{(1,0,2u_0), (0,1,2v_0)\big\}.$$

This is a two-dimensional subspace of $\mathbb{R}^3$, namely the tangent plane to S at the point $\mathbf{x}(u_0,v_0)$. Equivalently, one checks that every vector

$$(X,Y,Z) \in \operatorname{span}\{(1,0,2u_0), (0,1,2v_0)\}$$

satisfies $Z = 2u_0X + 2v_0Y$, which is exactly the equation of the tangent plane to the paraboloid $z = x^2+y^2$ at the point $(x,y,z) = (u_0, v_0, u_0^2+v_0^2)$. In this way, one sees concretely that

$$d\mathbf{x}_{(u_0,v_0)}\big(T_{(u_0,v_0)}\mathbb{R}^2\big) = \underbrace{\big\{(X,Y,Z) \in \mathbb{R}^3 : Z = 2u_0X + 2v_0Y\big\}}_{\text{tangent plane to } S \text{ at } (u_0,v_0,u_0^2+v_0^2)}.$$

The coordinates of a vector $w \in T_p(S)$ in the basis associated to a parametrisation $\mathbf{x}$ are determined as follows. Let $w = \alpha'(0)$ be the velocity vector of a curve $\alpha\colon (-\varepsilon,\varepsilon) \to S$ passing through p at $t=0$. Since $\mathbf{x}$ is a parametrisation of S, we can write

$$\alpha(t) = \mathbf{x}\big(u(t), v(t)\big),$$

where

$$\beta(t) = \big(u(t), v(t)\big)\colon (-\varepsilon,\varepsilon) \to U$$

satisfies $\beta(0) = q = \mathbf{x}^{-1}(p)$. Therefore,

$$\alpha'(0) = \frac{d}{dt}\Big(\mathbf{x}\circ\beta\Big)\Big|_{t=0} = \frac{d}{dt}\mathbf{x}\big(u(t),v(t)\big)\Big|_{t=0} = \mathbf{x}_u(q)u'(0) + \mathbf{x}_v(q)v'(0) = w.$$

Thus, in the basis $\{\mathbf{x}_u(q), \mathbf{x}_v(q)\}$, the vector w has coordinates $\big(u'(0), v'(0)\big)$, where $(u(t),v(t))$ is any parametrised curve in the (u,v)–plane whose image by $\mathbf{x}$ has velocity w at $t=0$.

With the notion of a tangent plane in hand, we can define the differential of a differentiable map between surfaces. Let S_1 and S_2 be two regular surfaces, and let

$$\varphi\colon V \subseteq S_1 \to S_2 \quad \text{be} \quad \text{a differentiable map of an open set } V \subseteq S_1 \text{ into } S_2.$$

Fix a point $p \in V$. Every tangent vector

$$w \in T_p(S_1)$$

can be realized as the velocity $\alpha'(0)$ of some differentiable curve

$$\alpha\colon (-\varepsilon,\varepsilon) \to V \quad \text{with} \quad \alpha(0) = p.$$

Define the curve $\beta(t) = \varphi\big(\alpha(t)\big)$. Since $\beta(0) = \varphi(p)$, its velocity $\beta'(0)$ is a vector in $T_{\varphi(p)}\big(S_2\big)$. One checks that $\beta'(0)$ depends only on $w = \alpha'(0)$ and not on the particular choice of α. Hence, we have the following proposition:

Proposition 2.7. In the discussion above, given $w \in T_p(S_1)$, the vector $\beta'(0) \in T_{\varphi(p)}(S_2)$ does not depend on the choice of curve α representing w. The map

$$d\varphi_p : T_p(S_1) \to T_{\varphi(p)}(S_2) \quad \text{where} \quad d\varphi_p(w) = \beta'(0)$$

is linear.

The linear map $d\varphi_p$ is called the differential of φ at $p \in S_1$. In a similar way, if

$$f\colon U \subseteq S \to \mathbb{R}$$

is a differentiable real–valued function on a regular surface S, one defines the differential at $p \in U$ as a linear map

$$df_p : T_p(S) \to \mathbb{R},$$

given by $df_p(w) = \frac{d}{dt}(f \circ \alpha)(t)\big|_{t=0}$ for any curve $\alpha(t)$ with $\alpha(0) = p$ and $\alpha'(0) = w$. We leave the straightforward details to the reader.

Example 2.11 (height function on a surface)**.** Let $v \in \mathbb{R}^3$ be a fixed unit vector, and define

$$h\colon S \to \mathbb{R} \quad h(p) = v \cdot p \quad \text{where } p \in S,$$

to be the *height function* of S in the direction v. To compute the differential dh_p on a tangent vector $w \in T_p(S)$, choose a differentiable curve

$$\alpha\colon (-\varepsilon,\varepsilon) \to S \quad \text{with} \quad \alpha(0) = p, \quad \alpha'(0) = w.$$

Since $h(\alpha(t)) = \alpha(t) \cdot v$, it follows that

$$dh_p(w) = \frac{d}{dt}\Big(h(\alpha(t))\Big)\Big|_{t=0} = \frac{d}{dt}\big(\alpha(t) \cdot v\big)\Big|_{t=0} = \alpha'(0) \cdot v = w \cdot v.$$

Example 2.12 (Rotation of the unit sphere)**.** Let

$$S^2 = \big\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\big\}$$

be the unit sphere in $\mathbb{R}^3$. For a fixed angle θ, let

$$R_{z,\theta}\colon \mathbb{R}^3 \to \mathbb{R}^3$$

be the rotation about the z–axis by angle θ. Restricted to the sphere,

$$R_{z,\theta}\big|_{S^2} : S^2 \to S^2$$

is a differentiable map. We compute its differential

$$\big(dR_{z,\theta}\big)_p : T_p(S^2) \to T_{R_{z,\theta}(p)}(S^2).$$

Fix $p \in S^2$ and let $w \in T_p(S^2)$. Choose a differentiable curve

$$\alpha\colon (-\varepsilon,\varepsilon)^2 \quad \text{with} \quad \alpha(0) = p, \quad \alpha'(0) = w.$$

Since $R_{z,\theta}$ is a linear map on $\mathbb{R}^3$, we have

$$\left(dR_{z,\theta}\right)_p(w) = \frac{d}{dt}\Big(R_{z,\theta}\circ\alpha\Big)(t)\Big|_{t=0} = R_{z,\theta}\left(\alpha'(0)\right) = R_{z,\theta}(w).$$

In particular, $R_{z,\theta}$ leaves the north pole

$$N = (0,0,1)$$

fixed, and thus

$$\left(dR_{z,\theta}\right)_N : T_N(S^2) \to T_N(S^2)$$

is simply the rotation of angle θ in the tangent plane $T_N(S^2)$.

In retrospect, what we have been doing up to now is extending the notions of Differential Calculus in $\mathbb{R}^2$ to regular surfaces. Since Calculus is essentially a local theory, we defined an object (the regular surface) which, locally, is diffeomorphic to a plane (Definition 2.7). It is natural, then, that the inverse function theorem in $\mathbb{R}^2$ should extend to maps between surfaces.

Definition 2.7 (local diffeomorphism)**.** Let $\varphi\colon U \subseteq S_1 \to S_2$ be a differentiable map between regular surfaces. We say that φ is a *local diffeomorphism* at a point $p \in U$ if there exists a neighborhood $V \subset U$ of p such that

$$\varphi\big|_V : V \to \varphi(V)$$

is a diffeomorphism onto an open set $\varphi(V) \subset S_2$.

Proposition 2.8. Let S_1 and S_2 be regular surfaces, and let

$$\varphi\colon U \subseteq S_1 \to S_2$$

be a differentiable mapping of an open set $U \subset S_1$ into S_2. If at a point $p \in U$ the differential

$$d\varphi_p : T_p(S_1) \to T_{\varphi(p)}(S_2)$$

is an isomorphism of tangent spaces, then φ is a local diffeomorphism at p.

Of course, all other standard concepts of Calculus such as critical points, regular values, etc. extend naturally to functions and maps on regular surfaces. The tangent plane also allows us to speak of the angle between two intersecting surfaces at a point of intersection. Given a point p on a regular surface S, there are exactly two unit vectors in $\mathbb{R}^3$ that are orthogonal to the tangent plane $T_p(S)$; each is called a unit normal vector at p. The straight line through p in the direction of a unit normal at p is called the normal line at p. The angle between two intersecting surfaces at their intersection point p is defined to be the angle between their corresponding tangent planes (or, equivalently, between their normal lines) at p.

By fixing one parametrization

$$\mathbf{x}\colon U \subseteq \mathbb{R}^2 \to S$$

at $p \in S$, one can make a consistent choice of unit normal vector at each point $q \in \mathbf{x}(U)$ by setting

$$N(q) = \frac{\mathbf{x}_u(q)\times\mathbf{x}_v(q)}{\left|\mathbf{x}_u(q)\times\mathbf{x}_v(q)\right|}.$$

Hence one obtains a smooth map

$$N\colon \mathbf{x}(U) \to \mathbb{R}^3.$$

We make some remarks regarding differentiability. The definition of a regular surface requires that each parametrisation be of class $\mathcal{C}^\infty$, i.e. all partial derivatives of all orders exist and are continuous. In practice, many results in Differential Geometry only require partials up to a finite order (often two or three, and very rarely more than four).

For instance, the existence and continuity of the tangent plane at every point depend only on the existence and continuity of the first partial derivatives. Consequently, one could have a surface given as the graph $z = f(x,y)$ of a function f with continuous first partials everywhere (so that a tangent plane exists at every point) but for which higher partials fail to exist or be continuous, violating the smoothness required of a regular surface. The following example illustrates this phenomenon.

Example 2.13. Consider the surface given by

$$z = \left(x^2+y^2\right)^{2/3}$$

which is obtained by rotating the planar curve $z = x^{4/3}$ about the z–axis. Since the generating curve $z = x^{4/3}$ has a continuous derivative that vanishes at the origin, the surface $z = \left(x^2+y^2\right)^{2/3}$ admits the xy–plane as a tangent plane at the origin. However, the second partial derivative $\frac{\partial^2 z}{\partial x^2}$ does not exist at $(0,0)$. Therefore, this surface fails to be a regular surface in the $\mathcal{C}^\infty$ sense.

2.4 The First Fundamental Form

So far, we have looked at surfaces from the point of view of differentiability. Now, we will begin the study of further geometric structures carried by the surface. The most important of these is perhaps the first fundamental form, which we shall now describe. The natural inner product of $\mathbb{R}^3 \supseteq S$ induces on each tangent plane $T_p(S)$ of a regular surface S an inner product, to be denoted by $\langle \cdot,\cdot \rangle_p$. If $w_1, w_2 \in T_p(S) \subseteq \mathbb{R}^3$, then $\langle w_1, w_2\rangle_p$ is equal to the inner product of w_1 and w_2 as vectors in $\mathbb{R}^3$. To this inner product, which is a symmetric bilinear form ($\langle w_1, w_2\rangle = \langle w_2, w_1\rangle$ and $\langle w_1, w_2\rangle$ is linear in both w_1 and w_2), there corresponds a quadratic form $I_p : T_p(S) \to \mathbb{R}$ given by $I_p(w) = \langle w, w\rangle_p = |w|^2 \geq 0$.

Definition 2.8. The quadratic form I_p on $T_p(S)$ defined by

$$I_p : T_p(S) \to \mathbb{R} \quad \text{where} \quad I_p(w) = \langle w, w\rangle_p = |w|^2 \geq 0$$

is called the first fundamental form of the regular surface $S \subseteq \mathbb{R}^3$ at $p \in S$.

So, the first fundamental form is merely the expression of how the surface S inherits the natural inner product of $\mathbb{R}^3$. Geometrically, the first fundamental form allows us to make measurements on the surface (lengths of curves, angles of tangent vectors, areas of regions) without referring back to the ambient space $\mathbb{R}^3$ where the surface lies.

We now express the first fundamental form in terms of the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ associated to a parametrisation $\mathbf{x}(u,v)$ at p. Since a tangent vector $w \in T_p(S)$ is the tangent vector to a parametrised curve $\alpha(t) = \mathbf{x}(u(t), v(t))$, $t \in (-\varepsilon, \varepsilon)$, with $p = \alpha(0) = \mathbf{x}(u_0, v_0)$, we obtain

$$I_p(\alpha'(0)) = \langle \alpha'(0), \alpha'(0)\rangle_p = \langle \mathbf{x}_u u' + \mathbf{x}_v v', \mathbf{x}_u u' + \mathbf{x}_v v'\rangle_p = E(u')^2 + 2Fu'v' + G(v')^2,$$

where the values of the functions involved are computed for $t = 0$, and

$$E(u_0, v_0) = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p \quad F(u_0, v_0) = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p \quad G(u_0, v_0) = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p.$$

It is convenient to observe that

$$|\mathbf{x}_u \times \mathbf{x}_v|^2 + \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2 = |\mathbf{x}_u|^2 \cdot |\mathbf{x}_v|^2$$

which shows that the integrand of $A(R)$ can be written as

$$|\mathbf{x}_u \times \mathbf{x}_v| = \sqrt{EG - F^2}.$$

Example 2.14. Let us compute the area of the torus. Consider the parametrisation

$$\mathbf{x}(u,v) = ((a + r\cos u)\cos v,\ (a + r\cos u)\sin v,\ r\sin u) \quad 0 < u < 2\pi,\ 0 < v < 2\pi.$$

The coefficients of the first fundamental form are

$$E = r^2 \quad F = 0 \quad G = (r\cos u + a)^2.$$

Hence, $\sqrt{EG - F^2} = r(r\cos u + a)$. Let

$$Q_\varepsilon = \{(u,v) \in \mathbb{R}^2 : \varepsilon < u < 2\pi - \varepsilon, \varepsilon < v < 2\pi - \varepsilon\},$$

and define $R_\varepsilon = \mathbf{x}(Q_\varepsilon)$. Then,

$$A(R_\varepsilon) = \iint_{Q_\varepsilon} r(r\cos u + a)dudv.$$

Evaluating the integral, we have

$$A(R_\varepsilon) = \int_\varepsilon^{2\pi-\varepsilon} \left(\int_\varepsilon^{2\pi-\varepsilon} r(r\cos u + a)du \right) dv = r^2(2\pi - 2\varepsilon)(\sin(2\pi - \varepsilon) - \sin\varepsilon) + ra(2\pi - 2\varepsilon)^2.$$

Letting $\varepsilon \to 0$, we obtain the total area of the torus, which is

$$A(\mathbb{T}) = \lim_{\varepsilon \to 0} A(R_\varepsilon) = 4\pi^2 ra.$$

Chapter 3
The Gauss Map

3.1
Orientation and the Gauss Map of Surfaces

We shall begin by briefly reviewing the notion of orientation for surfaces. As we have seen, given a parametrisation $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ of a regular surface S at a point $p \in S$, we can choose a unit normal vector at each point of $\mathbf{x}(U)$ by the rule $N(q) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}(q)$, where $q \in \mathbf{x}(U)$.

Thus, we have a differentiable map $N : \mathbf{x}(U) \to \mathbb{R}^3$ that associates to each $q \in \mathbf{x}(U)$ a unit normal vector $N(q)$.

More generally, if $V \subseteq S$ is an open set in S and $N : V \to \mathbb{R}^3$ is a differentiable map which associates to each $q \in V$ a unit normal vector at q, we say that N is a differentiable field of unit normal vectors on V.

It is a striking fact that not all surfaces admit a differentiable field of unit normal vectors defined on the whole surface. For instance, on the Möbius strip, one cannot define such a field. This can be seen intuitively by going around once along the middle circle of the strip: after one turn, the vector field $\mathbf{N}$ would come back as $-\mathbf{N}$, a contradiction to the continuity of $\mathbf{N}$. We shall say that a regular surface is orientable if it admits a differentiable field of unit normal vectors defined on the whole surface; the choice of such a field $\mathbf{N}$ is called an orientation of S.

An orientation N on S induces an orientation on each tangent space $T_p(S)$, $p \in S$, as follows. Define a basis $\{v, w\} \subseteq T_p(S)$ to be positive if $\langle v \times w, N \rangle$ is positive. It is easily seen that the set of all positive bases of $T_p(S)$ defines an orientation for $T_p(S)$.

At this juncture, S will denote a regular orientable surface in which an orientation (i.e. a differentiable field of unit normal vectors $\mathbf{N}$) has been chosen; this will simply be called a surface S with an orientation N.

Definition 3.1 (Gauss map)**.** Let $S \subseteq \mathbb{R}^3$ be a surface with an orientation N. The map $N : S \to \mathbb{R}^3$ takes its values in the unit sphere S^2. The map $N : S \to S^2$, thus defined, is called the Gauss map of S.

It is straightforward to verify that the Gauss map is differentiable. The differential dN_p of N at $p \in S$ is a linear map from $T_p(S)$ to $T_{N(p)}(S^2)$. Since $T_p(S)$ and $T_{N(p)}(S^2)$ are the same vector spaces, dN_p can be regarded as a linear map on $T_p(S)$.

The linear map $dN_p : T_p(S) \to T_p(S)$ operates as follows. For each parametrised curve $\alpha(t)$ in S with $\alpha(0) = p$, we consider the parametrised curve $N \circ \alpha(t) = N(t)$ in the sphere S^2. This amounts to restricting the normal vector N to the curve $\alpha(t)$. The tangent vector $N'(0) = dN_p(\alpha'(0))$ is a vector in $T_p(S)$. It measures the rate of change of the normal vector N, restricted to the curve $\alpha(t)$, at $t = 0$. Thus, dN_p measures how N pulls away from $N(p)$ in a neighbourhood of p. In the case of curves, this measure is given by a number (the curvature); in the case of surfaces, it is characterised by a linear map.

Example 3.1. For a plane P given by $ax+by+cz+d=0$, the unit normal vector $\mathbf{N} = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}$ is constant, and therefore $dN \equiv 0$.

Example 3.2. Consider the unit sphere S^2. Let $\alpha(t) = (x(t), y(t), z(t))$ be a parametrised curve in S^2. Then $2xx' + 2yy' + 2zz' = 0$. This shows that the vector (x,y,z) is normal to the sphere at the point (x,y,z). Thus, $\overline{\mathbf{N}} = (x,y,z)$ and $\mathbf{N} = (-x,-y,-z)$ are fields of unit normal vectors in S^2. We fix an orientation in S^2 by choosing $\mathbf{N} = (-x,-y,-z)$ as a normal field. Notice that $\mathbf{N}$ points away from the centre of the sphere.

Restricted to the curve $\alpha(t)$, the normal vector is: $\mathbf{N}(t) = (-x(t), -y(t), -z(t))$. Therefore,

$$d\mathbf{N}(x'(t), y'(t), z'(t)) = \mathbf{N}'(t) = (-x'(t), -y'(t), -z'(t)).$$

That is,

$$dN_p(v) = -v \quad \text{for all } p \in S^2 \text{ and } v \in T_p(S^2).$$

If we had chosen $\overline{\mathbf{N}}$ instead, we would have obtained $d\overline{\mathbf{N}}_p(v) = v$.

Example 3.3. Consider the cylinder $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$. A similar argument gives the normal vector $\mathbf{N} = (-x,-y,0)$. For a curve $(x(t), y(t), z(t))$ contained in the cylinder,

$$\mathbf{N}(t) = (-x(t), -y(t), 0) \quad \text{and} \quad d\mathbf{N}(x'(t), y'(t), z'(t)) = \mathbf{N}'(t) = (-x'(t), -y'(t), 0).$$

If v is tangent to the cylinder and parallel to the z-axis, then $dN(v) = 0 = 0v$; if w is tangent to the cylinder and parallel to the xy-plane, then $dN(w) = -w$. Hence, v and w are eigenvectors of dN with eigenvalues 0 and -1, respectively. Normal curvatures vary from 0 to 1 across a family of ellipses. Eigenvectors v and w of dN_p have eigenvalues 0 and -1. The extreme values of the second fundamental form occur at these directions.

Example 3.4. Consider the surface of revolution obtained by rotating the curve $z = y^4$ about the z-axis. At $p = (0,0,0)$, the differential $dN_p = 0$ since the curvature of $z = y^4$ at p is zero and $N(p)$ is parallel to the z-axis. Thus, all normal curvatures are zero at p.

Example 3.5. For the plane $ax+by+cz+d=0$, all normal curvatures are zero, so $d\mathbf{N} \equiv 0$. For the sphere S^2, normal sections are circles of radius 1. So, all normal curvatures equal 1, and $\Pi_p(v) = 1$ for all unit $\mathbf{v} \in T_p(S)$.

Let us now return to the linear map dN_p.

Definition 3.2 (maximum and minimum normal curvature)**.** The maximum normal curvature k_1 and the minimum normal curvature k_2 are principal curvatures at p; their corresponding directions, i.e., eigenvectors e_1, e_2, are called principal directions.

For instance, in a plane or sphere, all directions are principal directions.

Definition 3.3 (line of curvature)**.** If a regular connected curve $C \subseteq S$ is such that for all $p \in C$, the tangent of C at p is a principal direction, then C is a line of curvature.

Proposition 3.1 (Rodrigues)**.** A necessary and sufficient condition for $C \subseteq S$ to be a line of curvature is that

$$N'(t) = \lambda(t)\alpha'(t) \quad \text{for a parametrisation } \alpha(t) \text{ of } C \text{ where } \lambda(t) \text{ is a differentiable function.}$$

In this case, $-\lambda(t)$ is the (principal) curvature.

Given $v = e_1 \cos\theta + e_2 \sin\theta$, the normal curvature is

$$k_n = \mathrm{II}_p(v) = -\langle dN_p(v), v\rangle = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

This is Euler's formula.

Given a linear map A on a 2D space with basis $\{v_1, v_2\}$, we recall that:

$$\det A = a_{11}a_{22} - a_{12}a_{21} \quad \text{and} \quad \operatorname{tr} A = a_{11} + a_{22}.$$

Definition 3.4. Let $p \in S$, and let dN_p be the differential of the Gauss map. Then, we have the following:

- The determinant of dN_p is the Gaussian curvature K
- The negative half of the trace of dN_p is the mean curvature H

With principal curvatures k_1, k_2, we have $K = k_1k_2$ and $H = \frac{k_1+k_2}{2}$.

Definition 3.5. A point $p \in S$ is:

- Elliptic if $\det(dN_p) > 0$
- Hyperbolic if $\det(dN_p) < 0$
- Parabolic if $\det(dN_p) = 0$, but $dN_p \neq 0$
- Planar if $dN_p = 0$

Definition 3.6. If at $p \in S$, $k_1 = k_2$, then p is called an umbilical point. Planar points are umbilical with $k_1 = k_2 = 0$.

Proposition 3.2. If all points of a connected surface S are umbilical, then S is either a plane or a sphere.

Definition 3.7 (asymptotic direction and curve)**.** An asymptotic direction of S at p is a direction in $T_p(S)$ where the normal curvature is zero. A curve whose tangents are asymptotic directions is called an asymptotic curve.

Let $\Pi_p(w) = \pm 1$. Using polar coordinates $w = \rho v$, we derive

$$\pm 1 = \rho^2 \Pi_p(v) = k_1\xi^2 + k_2\eta^2.$$

This is the Dupin indicatrix equation.

Definition 3.8. Two non-zero vectors $w_1, w_2 \in T_p(S)$ are conjugate if $\langle dN_p(w_1), w_2\rangle = \langle w_1, dN_p(w_2)\rangle = 0$.

Given an orthonormal basis $\{e_1, e_2\}$ with principal curvatures k_1, k_2, two directions r_1, r_2 with angles θ, φ are conjugate if and only if $k_1 \cos\theta \cos\varphi = -k_2 \sin\theta \sin\varphi$. This leads to the geometric construction of conjugate directions via the Dupin indicatrix.

3.2 The Gauss Map in Local Coordinates

Previously, we introduced some concepts related to the local behaviour of the Gauss map. To emphasise the geometry of the situation, the definitions were given without the use of a coordinate system. Some simple examples were then computed directly from the definitions; this procedure, however, is inefficient in handling general situations. In this section, we shall obtain the expressions of the second fundamental form and of the differential of the Gauss map in a coordinate system. This will give us a systematic method for computing specific examples. Moreover, the general expressions thus obtained are essential for a more detailed investigation of the concepts introduced previously.

Throughout this section, all parametrisations $\mathbf{x}: U \subseteq \mathbb{R}^2 \to S$ considered are assumed to be compatible with the orientation of S; that is, in $x(U)$,

$$N = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}.$$

Let $\mathbf{x}(u,v)$ be a parametrisation at a point $p \in S$ of a surface S, and let

$$\alpha(t) = \mathbf{x}\big(u(t), v(t)\big) \quad \alpha(0) = p.$$

To simplify notation, we shall make the convention that all functions to appear below denote their values at the point p. The tangent vector to $\alpha(t)$ at p is

$$\alpha'(t)\big|_{t=0} = \mathbf{x}_u u' + \mathbf{x}_v v'.$$

Since N_u and N_v belong to $T_p(S)$, we may write

$$\begin{aligned} N_u &= a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v \\ N_v &= a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v \end{aligned}$$

and therefore, for $\alpha'(0) = \mathbf{x}_u u' + \mathbf{x}_v v'$,

$$dN\big(\alpha'\big) = N_u u' + N_v v' = (a_{11}u' + a_{12}v')\mathbf{x}_u + (a_{21}u' + a_{22}v')\mathbf{x}_v.$$

Hence, in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$,

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}.$$

Note that this matrix $\big(a_{ij}\big)$ is not necessarily symmetric, unless $\{\mathbf{x}_u, \mathbf{x}_v\}$ is an orthonormal basis of $T_p(S)$. On the other hand, the expression of the second fundamental form in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ is given by

$$\Pi_p(\alpha', \alpha') = -\big\langle N_u u' + N_v v', \mathbf{x}_u u' + \mathbf{x}_v v'\big\rangle = e(u')^2 + 2fu'v' + g(v')^2,$$

where, since $\langle N, \mathbf{x}_u\rangle = \langle N, \mathbf{x}_v\rangle = 0$,

$$\begin{aligned} e &= -\langle N_u, \mathbf{x}_u\rangle = \langle N, \mathbf{x}_{uu}\rangle \\ f &= -\langle N_u, \mathbf{x}_v\rangle = -\langle N, \mathbf{x}_{uv}\rangle \\ g &= -\langle N_v, \mathbf{x}_v\rangle = \langle N, \mathbf{x}_{vv}\rangle \end{aligned}$$

We shall now obtain the values of a_{ij} in terms of the coefficients e, f, g. We have

$$\begin{aligned} -f &= _u, \mathbf{x}_v\rangle = a_{11}\langle \mathbf{x}_u, \mathbf{x}_v\rangle + a_{21}\langle \mathbf{x}_v, \mathbf{x}_v\rangle = a_{11}F + a_{21}G, \\ -f &= _v, \mathbf{x}_u\rangle = a_{12}\langle \mathbf{x}_u, \mathbf{x}_u\rangle + a_{22}\langle \mathbf{x}_u, \mathbf{x}_v\rangle = a_{12}E + a_{22}F, \\ -e &= _u, \mathbf{x}_u\rangle = a_{11}\langle \mathbf{x}_u, \mathbf{x}_u\rangle + a_{21}\langle \mathbf{x}_u, \mathbf{x}_v\rangle = a_{11}E + a_{21}F, \\ -g &= _v, \mathbf{x}_v\rangle = a_{12}\langle \mathbf{x}_u, \mathbf{x}_v\rangle + a_{22}\langle \mathbf{x}_v, \mathbf{x}_v\rangle = a_{12}F + a_{22}G, \end{aligned} \tag{3.1}$$

where

$$E = \langle \mathbf{x}_u, \mathbf{x}_u\rangle, F = \langle \mathbf{x}_u, \mathbf{x}_v\rangle, G = \langle \mathbf{x}_v, \mathbf{x}_v\rangle$$

are the coefficients of the first fundamental form in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$. Relations (3.1) may be expressed in matrix form as

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}. \tag{3.2}$$

Hence

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}.$$

Since

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG-F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix},$$

we obtain the *equations of Weingarten*:

$$\begin{aligned} a_{11} &= \frac{fF-eG}{EG-F^2}, & a_{12} &= \frac{gF-fG}{EG-F^2}, \\ a_{21} &= \frac{eF-fE}{EG-F^2}, & a_{22} &= \frac{fF-gE}{EG-F^2}. \end{aligned} \tag{3.3}$$

Gaussian and mean curvature in coordinates. From (**??**) and (3.3), we immediately obtain the *Gaussian curvature*:

$$K = \det(a_{ij}) = \frac{eg-f^2}{EG-F^2}. \tag{3.4}$$

To compute the *mean curvature*, recall that if $-k_1, -k_2$ are the eigenvalues of the linear map dN on $T_p(S)$, then k_1, k_2 are the principal curvatures. Thus, for some nonzero $v \in T_p(S)$,

$$dN(v) = -kv.$$

Since I is the identity map on $T_p(S)$, the linear map $dN + kI$ is not invertible; hence it has zero determinant. That is,

$$\det(a_{ij} + k\delta_{ij}) = 0, \qquad \iff \qquad \begin{vmatrix} a_{11}+k & a_{12} \\ a_{21} & a_{22}+k \end{vmatrix} = 0,$$

or equivalently,

$$k^2 + k(a_{11}+a_{22}) + a_{11}a_{22} - a_{12}a_{21} = 0.$$

Since k_1 and k_2 are the roots of this quadratic, the *mean curvature*

$$H = \tfrac{1}{2}(k_1+k_2) = -\tfrac{1}{2}(a_{11}+a_{22}).$$

Using (3.3), one checks that

$$H = -\tfrac{1}{2}(a_{11}+a_{22}) = \frac{Eg-2Ff+Ge}{2(EG-F^2)}. \tag{3.5}$$

Moreover, the principal curvatures satisfy

$$k^2 - 2Hk + K = 0, \quad \text{so that} \quad k = H \pm \sqrt{H^2-K}.$$

If we choose $k_1(p) \geq k_2(p)$ at each point $p \in S$, then the functions k_1 and k_2 are continuous on S. Moreover, k_1 and k_2 are differentiable on S, except perhaps at the umbilical points (where $H^2 = K$).

In computations in this chapter, it will be convenient to write, for any vectors $u, v, w \in \mathbb{R}^3$,

$$(u \wedge v, w) = \det \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix},$$

noting that this is merely the determinant of the 3×3 matrix whose columns (or rows) are the components of u, v, w in the canonical basis of $\mathbb{R}^3$.

Example 3.6 (Torus of Revolution)**.** Consider the torus obtained by revolving a circle of radius r in the plane about an axis lying in that plane at distance $a > r$ from the center. A convenient parametrization is

$$\mathbf{x}(u,v) = \big((a+r\cos u)\cos v, (a+r\cos u)\sin v, r\sin u\big), \quad 0 < u < 2\pi, \quad 0 < v < 2\pi.$$

For the computation of the coefficients e, f, g, we need to know N (and thus $\mathbf{x}_u$ and $\mathbf{x}_v$), $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, and $\mathbf{x}_{vv}$:

$$\begin{aligned}
\mathbf{x}_u &= \big(-r\sin u\cos v, -r\sin u\sin v, r\cos u\big),\\
\mathbf{x}_v &= \big(-(a+r\cos u)\sin v, (a+r\cos u)\cos v, 0\big),\\
\mathbf{x}_{uu} &= \big(-r\cos u\cos v, -r\cos u\sin v, -r\sin u\big),\\
\mathbf{x}_{uv} &= \big(r\sin u\sin v, -r\sin u\cos v, 0\big),\\
\mathbf{x}_{vv} &= \big(-(a+r\cos u)\cos v, -(a+r\cos u)\sin v, 0\big).
\end{aligned}$$

From these, we obtain the coefficients of the first fundamental form:

$$E = \langle \mathbf{x}_u, \mathbf{x}_u\rangle = r^2, \qquad F = \langle \mathbf{x}_u, \mathbf{x}_v\rangle = 0, \qquad G = \langle \mathbf{x}_v, \mathbf{x}_v\rangle = (a+r\cos u)^2.$$

Since

$$\big\|\mathbf{x}_u \times \mathbf{x}_v\big\| = \sqrt{EG - F^2} = r\big(a + r\cos u\big),$$

the unit normal is

$$N = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}.$$

Introducing the values just obtained into $e =, \mathbf{x}_{uu}\rangle$, we have

$$e = \frac{\big(\mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{uu}\big)}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{r^2(a+r\cos u)}{\sqrt{EG-F^2}} = \frac{r^2(a+r\cos u)}{r(a+r\cos u)} = r.$$

Similarly, one computes

$$f = \frac{\big(\mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{uv}\big)}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = 0, \qquad g = \frac{\big(\mathbf{x}_u \times \mathbf{x}_v, \mathbf{x}_{vv}\big)}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \cos u\big(a + r\cos u\big).$$

Finally, since

$$K = \frac{eg - f^2}{EG - F^2},$$

we have

$$K = \frac{r\cdot\big[\cos u(a+r\cos u)\big]}{r^2(a+r\cos u)^2} = \frac{\cos u}{r\big(a+r\cos u\big)}.$$

From this expression, it follows that $K = 0$ along the parallels $u = \pi/2$ and $u = 3\pi/2$; the points of such parallels are therefore *parabolic* points. In the region of the torus given by $\pi/2 < u < 3\pi/2$, K is negative (notice that $r > 0$ and $a > r$); the points in this region are therefore *hyperbolic* points. In the region given by $0 < u < \pi/2$ or $3\pi/2 < u < 2\pi$, the curvature is positive and the points are *elliptic* points (see Figure 3–15).

As an application of the expression for the second fundamental form in coordinates, we shall prove a proposition which gives information about the position of a surface in the neighbourhood of an elliptic or a hyperbolic point, relative to the tangent plane at this point.

Proposition 3.3. Let $p \in S$ be an elliptic point of a surface S. Then there exists a neighbourhood V of p in S such that all points in V belong to the same side of the tangent plane $T_p(S)$. Let $p \in S$ be a hyperbolic point. Then in each neighbourhood of p there exist points of S in both sides of $T_p(S)$.

No such statement as Proposition **??** can be made in a neighborhood of a parabolic or planar point. In the above examples of parabolic and planar points (cf. Examples 3 and 6 of Sec.3–2) the surface lies on one side of the tangent plane and may have a line in common with this plane. In the following examples we shall show that an entirely different situation may occur.

Example 3.7 (The "Monkey Saddle"). The "monkey saddle" (see Fig. 3–17) is given by

$$x = u, \qquad y = v, \qquad z = u^3 - 3uv^2.$$

A direct computation shows that at $(0,0,0)$ the coefficients of the second fundamental form are

$$e = f = g = 0;$$

the point $(0,0,0)$ is therefore a planar point. In any neighborhood of this point, however, there are points in both sides of its tangent plane.

Example 3.8 (Rotational Parabolic Points). Consider the surface obtained by rotating the curve $z = y^3, -1 < z < 1$, about the z-axis. A simple computation shows that the points generated by the rotation of the origin O are *parabolic* points. We shall omit this computation, because we shall prove shortly (Example 4 of Sec.3–2) that the parallels (curves of the form $y = x$) have zero curvature and the meridian section (curves of the form $y = x^3$) also have zero curvature. Notice that in any neighborhood of such a parabolic point there exist points of S in both sides of the tangent plane.

Asymptotic directions and principal directions. The expression of the second fundamental form in local coordinates is particularly useful for the study of the asymptotic and principal directions. Let us first look at the asymptotic directions.

Let $\mathbf{x}(u,v)$ be a parametrization at $p \in S$, with $\mathbf{x}(0,0) = p$, and let $e(u,v), f(u,v)$, and $g(u,v)$ be the coefficients of the second fundamental form in this parametrization. We recall that (see Def.9 of Sec.3–2) a connected regular curve C in the coordinate neighborhood of $\mathbf{x}$ is an *asymptotic* curve if and only if for any parametrization $\alpha(t) = \mathbf{x}\big(u(t), v(t)\big), t \in I$, of C, we have $II\big(\alpha'(t), \alpha'(t)\big) = 0$, for all $t \in I$, that is, if and only if

$$e(u')^2 + 2fu'v' + g(v')^2 = 0, \quad t \in I. \tag{3.6}$$

Because of that, Eq.(3.6) is called the *differential equation of the asymptotic curves*. For the time being, we want to draw from Eq.(3.6) only the following useful conclusion: A necessary and sufficient condition for a parametrization in a neighborhood of a hyperbolic point is that

$$ef^2 < 0$$

be such that the coordinate curves $u = \text{const.}$, $v = v(t)$ and $u = u(t)$, $v = \text{const.}$ satisfy Eq.(3.6). Conversely, if this last condition holds and $f \neq 0$, Eq.(3.6) becomes $fu'v' = 0$, which is clearly satisfied by the coordinate lines.

We shall now consider the *principal directions*, maintaining the notations already established. A connected regular curve C in the coordinate neighborhood of $\mathbf{x}$ is a *line of curvature* if and only if for any parametrization $\alpha(t) = \mathbf{x}\big(u(t), v(t)\big)$ of C, $t \in I$, we have $dN\big(\alpha'(t)\big) = \lambda(t)\alpha'(t)$ for $t \in I$ (cf. Prop.3 of Sec.3–2).

It follows that the functions $u'(t), v'(t)$ satisfy the system of equations

$$\begin{aligned} \big(fF - eG\big)u' + \big(fG - Fg\big)v' &=', \\ \big(Eg - fF\big)u' + \big(fG - eF\big)v' &=' . \end{aligned} \tag{3.7}$$

By eliminating λ in the above system, we obtain the *differential equation of the lines of curvature*:

$$\big(Eg-eF\big)(u')^2+\big(eG-Eg\big)u'v'+\big(fG-Fg\big)(v')^2=0, \tag{3.8}$$

which may be written, in a more symmetric way, as

$$\begin{vmatrix} (u')^2 & -u'v' & (v')^2 \\ E & F & G \\ e & f & g \end{vmatrix}=0.$$

Example 3.9 (Surfaces of Revolution). Consider a surface of revolution parametrized by (cf. Example 4 of Sec.2–3; we have replaced v and u by ψ and φ, respectively):

$$\mathbf{x}(u,v)=\big(\varphi(v)\cos u,\varphi(v)\sin u,\psi(v)\big),\quad 0<u<2\pi,\quad a<v<b,\quad \varphi'(v)\neq 0.$$

The coefficients of the first fundamental form are given by

$$E=\varphi^2,\qquad F=0,\qquad G=\big(\varphi'\big)^2+\big(\psi'\big)^2.$$

It is convenient to assume that the rotating curve is parametrized by arc length, that is,

$$\big(\varphi'(v)\big)^2+\big(\psi'(v)\big)^2=G=1.$$

The computation of the coefficients of the second fundamental form is straightforward and yields

$$e=\frac{(\mathbf{x}_u\times\mathbf{x}_v,\mathbf{x}_{uu})}{\sqrt{EG-F^2}}=\frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} -\varphi\sin u & \varphi'(v)\cos u & -\varphi\cos u \\ \varphi\cos u & \varphi'(v)\sin u & -\varphi\sin u \\ 0 & \psi'(v) & 0 \end{vmatrix}=-\varphi\psi',$$

$$f=0,\qquad g=\psi'\varphi''-\varphi'\psi'.$$

Since $F=f=0$, we conclude that the parallels ($v=$ const.) and the meridians ($u=$ const.) of a surface of revolution are lines of curvature of such a surface (this fact was used in Example 3.8).

Because

$$K=\frac{eg-f^2}{EG-F^2}=\frac{-\varphi\psi'\big(\psi'\varphi''-\varphi'\psi'\big)}{\varphi^2\cdot 1}=-\frac{\psi'\big(\psi'\varphi''-\varphi'\psi'\big)}{\varphi},$$

and $\varphi(v)$ is always positive, it follows that the parabolic points are given by either

$\psi'=0$ (the tangent line to the generator curve is perpendicular to the axis of rotation) or $\varphi'\psi''-\psi'\varphi''=0$ (the curva

A point which satisfies both conditions is a planar point, since these conditions imply $e=f=g=0$.

It is convenient to put the Gaussian curvature in still another form. By differentiating

$$\big(\varphi'\big)^2+\big(\psi'\big)^2=1\quad\Longrightarrow\quad \varphi'(v)\varphi''(v)+\psi'(v)\psi''(v)=0,$$

we obtain $\varphi'\varphi''=-\psi'\psi''$. Thus,

$$K=-\frac{\psi'}{\varphi}\big(\psi'\varphi''-\varphi'\psi'\big)=-\frac{\psi'}{\varphi}\Big(\psi'\varphi''+\psi''\varphi'\Big)=-\frac{\varphi''}{\varphi}.$$

Equation above is a convenient expression for the Gaussian curvature of a surface of revolution. It can be used, for instance, to determine the surfaces of revolution of constant Gaussian curvature (cf. Exercise 7).

To compute the principal curvatures, we first make the following general observation: If a parametrization of a regular surface is such that $F=f=0$, then the coordinate curves $u=$ const. and $v=$ const. are lines of curvature. In particular, for a surface of revolution with $F=f=0$, the parallels and meridians are lines of curvature.

From equations (4) and (5), we have the expressions for the Gaussian and mean curvatures:

$$K = \frac{eg - f^2}{EG - F^2}, \qquad H = \frac{1}{2} \cdot \frac{eG - 2Ff + gE}{EG - F^2}.$$

If $F = f = 0$, the principal curvatures are simply $k_1 = \frac{e}{E}$, $k_2 = \frac{g}{G}$. Thus, for surfaces of revolution, we obtain:

$$\frac{e}{E} = -\frac{\psi'\varphi'}{\varphi^2}, \qquad \frac{g}{G} = \frac{-\varphi''\psi' + \psi''\varphi'}{\varphi}. \tag{3.9}$$

Hence, the mean curvature becomes:

$$H = \frac{1}{2}\left(-\frac{\psi'}{\varphi} + \frac{\psi'\varphi'' - \psi''\varphi'}{\varphi}\right) = \frac{1}{2} \cdot \frac{-\psi' + \varphi(\psi'\varphi'' - \psi''\varphi')}{\varphi}. \tag{3.10}$$

Example 5. Consider a surface given as the graph of a differentiable function $z = h(x,y)$. We can parametrize the surface as

$$\mathbf{x}(u,v) = (u, v, h(u,v)), \quad u = x, v = y.$$

We compute the partial derivatives:

$$\mathbf{x}_u = (1, 0, h_u), \qquad \mathbf{x}_v = (0, 1, h_v),$$
$$\mathbf{x}_{uu} = (0, 0, h_{uu}), \quad \mathbf{x}_{vv} = (0, 0, h_{vv}), \quad \mathbf{x}_{uv} = (0, 0, h_{uv}).$$

The normal vector field is

$$N(x,y) = \frac{(-h_x, -h_y, 1)}{(1 + h_x^2 + h_y^2)^{1/2}},$$

and the coefficients of the second fundamental form are:

$$e = \frac{h_{xx}}{(1 + h_x^2 + h_y^2)^{1/2}}, \quad f = \frac{h_{xy}}{(1 + h_x^2 + h_y^2)^{1/2}}, \quad g = \frac{h_{yy}}{(1 + h_x^2 + h_y^2)^{1/2}}.$$

From here we obtain:

$$K = \frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2},$$
$$2H = \frac{(1 + h_y^2)h_{xx} - 2h_x h_y h_{xy} + (1 + h_x^2)h_{yy}}{(1 + h_x^2 + h_y^2)^{3/2}}.$$

Dupin Indicatrix and Geometric Interpretation. Locally, a surface near a non-planar point p can be expressed as $z = h(x,y)$, where

$$h(0,0) = 0, \quad h_x(0,0) = h_y(0,0) = 0, \quad h_{xx}(0,0) = a, \quad h_{xy}(0,0) = 0, \quad h_{yy}(0,0) = b.$$

Then the surface can be approximated as:

$$h(x,y) \approx \frac{1}{2}(ax^2 + 2cxy + by^2).$$

The curve $C = \{(x,y) \in T_p(S) : h(x,y) = \varepsilon\}$ is approximately the curve

$$k_1 x^2 + k_2 y^2 = 2\varepsilon.$$

Scaling with similarity transformation gives the Dupin indicatrix:

$$k_1 \bar{x}^2 + k_2 \bar{y}^2 = 1.$$

Proposition 2. Let p be a point of a surface S with $K(p) \neq 0$, and let V be a connected neighborhood where K does not change sign. Then:

$$K(p) = \lim_{A \to 0} \frac{A'}{A},$$

where A is the area of a region $B \subseteq V$ containing p, and A' is the area of the image of B by the Gauss map $N\colon S \to S^2$.

Remark. Compare this with the curvature of a plane curve C at a point p:

$$k = \lim_{s \to 0} \frac{\sigma}{s},$$

where s is arc length of C near p, and σ is the arc length of its image in the indicatrix of tangents. This highlights that the Gaussian curvature K for surfaces is analogous to curvature k of plane curves.

Chapter 4
The Geometry of Surfaces